

SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY (s, m)-CONVEX FUNCTIONS IN SECOND SENSE

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ABSTRACT. Authors introduce the concept of harmonically (s, m)-convex functions in second sense in [1]. In this article, we establish some Hermite-Hadamard type inequalities of this class of functions.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then following inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

holds. This double inequality is known in the literature as the Hermite-Hadamard integral inequality for convex function. Note that some of the classical inequalities for mean can be derived from (1.1) for the appropriate particular selection of mapping f . Both inequalities in (1.1) hold in the reverse direction if f is concave.

In [4], İmdat İşcan introduced the concept of harmonically convex function, and established a variant of Hermite-Hadamard type inequalities which holds for these classes of functions as follows:

Definition 1.1. Let $I \subset \mathbb{R}/\{0\}$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$(1.2) \quad f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in I$ and $t \in [0, 1]$. If inequality in (1.2) is reversed, then f is said to be harmonically concave.

Theorem 1.2. (see [4]) Let $f : I \subset \mathbb{R}/\{0\} \rightarrow \mathbb{R}$ be harmonically convex and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then following inequalities hold

$$(1.3) \quad f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.$$

Theorem 1.3. (see [4]). Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable on I° , $a, b \in I$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q \geq 1$, then

$$(1.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \lambda_1^{1-\frac{1}{q}} (\lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q)^{\frac{1}{q}},$$

where

$$\begin{aligned} \lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left(\frac{(a+b)^2}{4ab} \right), \\ \lambda_2 &= \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left(\frac{(a+b)^2}{4ab} \right), \\ \lambda_3 &= \frac{1}{a(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left(\frac{(a+b)^2}{4ab} \right). \end{aligned}$$

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Theorem 1.4. (see [4]). Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$ and $f' \in [a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(1.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} (\mu_1 |f'(a)|^q + \mu_2 |f'(b)|^q)^{\frac{1}{q}},$$

where

$$\mu_1 = \frac{[a^{2-2q} + b^{1-2q}[(b-a)(1-2q) - a]]}{2(b-a)^2(1-q)(1-2q)},$$

$$\mu_2 = \frac{[b^{2-2q} - a^{1-2q}[(b-a)(1-2q) + b]]}{2(b-a)^2(1-q)(1-2q)}.$$

In [5], Imdat Iscan introduced the concept of harmonically s -convex function in second sense as follow:

Definition 1.5. A function $f : I \subset \mathbb{R}/\{0\} \rightarrow \mathbb{R}$ is said to be harmonically s -convex in second sense, if

$$(1.6) \quad f\left(\frac{xy}{tx + (1-t)y}\right) \leq t^s f(y) + (1-t)^s f(x)$$

for all $x, y \in I$ and $t \in [0, 1]$. If inequality in (1.6) is reversed, then f is said to be harmonically s -concave.

Remark 1.6. Note that for $s = 1$, harmonically s -convexity reduces to ordinary harmonically convexity.

In [3], Feixiang Chen and Shanhe Wu generalized Hermite-Hadamard type inequalities given in [4] which hold for harmonically s -convex functions in second sense.

Theorem 1.7. (see [3]). Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically s -convex in second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $q \geq 1$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} C_1^{1-\frac{1}{q}}(a, b) [C_2(s; a, b) |f'(a)|^q + C_3(s; a, b) |f'(b)|^q]^{\frac{1}{q}},$$

where

$$C_1(a, b) = b^{-2} \left({}_2F_1(2, 2; 3, 1 - \frac{a}{b}) - {}_2F_1(2, 1; 2, 1 - \frac{a}{b}) + \frac{1}{2} {}_2F_1(2, 1; 3, \frac{1}{2}(1 - \frac{a}{b})) \right)$$

$$C_2(s, a, b) = b^{-2} \left(\frac{2}{s+2} {}_2F_1(2, s+2; s+3, 1 - \frac{a}{b}) - \frac{1}{s+1} {}_2F_1(2, s+1; s+2, 1 - \frac{a}{b}) \right. \\ \left. + \frac{1}{2^s(s+1)(s+2)} {}_2F_1(2, s+1; s+3, \frac{1}{2}(1 - \frac{a}{b})) \right)$$

$$C_3(s, a, b) = b^{-2} \left(\frac{2}{(s+1)(s+2)} {}_2F_1(2, 2; s+3, 1 - \frac{a}{b}) - \frac{1}{s+1} {}_2F_1(2, 1; s+2, 1 - \frac{a}{b}) \right. \\ \left. + \frac{1}{2} {}_2F_1(2, 1; 3, \frac{1}{2}(1 - \frac{a}{b})) \right)$$

Remark 1.8. Note that for $s = 1$, $C_1(a, b) = \lambda_1$, $C_2(1, a, b) = \lambda_2$ and $C_3(1, a, b) = \lambda_3$. Hence, Theorem 1.3 is particular case of theorem 1.7 for $s = 1$.

Theorem 1.9. (see [3]) Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically s -convex in second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $q > 1$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right|$$

$$\leq \frac{a(b-a)}{2b} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{1}{s+1} [{}_2F_1(2q, s+1; s+2, 1 - \frac{a}{b}) |f'(b)|^q + {}_2F_1(2q, 1; s+2, 1 - \frac{a}{b}) |f'(a)|^q] \right]^{\frac{1}{q}}$$

Remark 1.10. Note that for $s = 1$

$$\mu_1 = \frac{1}{2b^{2q}} \cdot {}_2F_1(2q, 2, 3, 1 - \frac{a}{b}),$$

and

$$\mu_2 = \frac{1}{2b^{2q}} \cdot {}_2F_1(2q, 1, 3, 1 - \frac{a}{b}).$$

Hence, Theorem 1.4 is particular case of theorem 1.9 for $s = 1$.

In ([7]), Jaekeun Park considered the class of (s, m) -convex functions in second sense. This class of function is defined as follow

Definition 1.11. For some fixed $s \in (0, 1]$ and $m \in [0, 1]$ a mapping $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ is said to be (s, m) -convex in the second sense on I if

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y)$$

holds, for all $x, y \in I$ and $t \in [0, 1]$

In ([6]), Imdat Iscan introduced the concept of harmonically (α, m) -convex functions and established some Hermite-Hadamard type inequalities for this class of function. This class of functions is defined as follow

Definition 1.12. The function $f : (0, \infty) \rightarrow \mathbb{R}$ is said to be harmonically (α, m) -convex, where $\alpha \in [0, 1]$ and $m \in (0, 1]$, if

$$(1.7) \quad \left(\frac{mxy}{mty + (1-t)x} \right) = f\left(\left(\frac{t}{x} + \frac{1-t}{my}\right)^{-1}\right) \leq t^\alpha f(x) + m(1-t)^\alpha f(y)$$

for all $x, y \in (0, \infty)$ and $t \in [0, 1]$. If the inequality in (1.7) is reversed, then f is said to be harmonically (α, m) -concave.

In [1], authors introduce the concept of Harmonically (s, m) -convex functions in second sense which generalize the notion of Harmonically convex and Harmonically s -convex functions in second sense introduced by Imdat Iscan in [4],[5].

In this paper, we establish some results connected with the right side of new inequality similar to (1.1) for this class of functions such that results given by Imdat Iscan [4], Feixiang Chen and Shanhe Wu [3] are obtained for the particular values of s, m .

Definition 1.13. The function $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ is said to be harmonically (s, m) -convex in second sense, where $s \in (0, 1]$ and $m \in (0, 1]$ if

$$f\left(\frac{mxy}{mty + (1-t)x}\right) = f\left(\left(\frac{t}{x} + \frac{1-t}{my}\right)^{-1}\right) \leq t^s f(x) + m(1-t)^s f(y)$$

$\forall x, y \in I$ and $t \in [0, 1]$.

Remark 1.14. Note that for $s = 1$, (s, m) -convexity reduces to harmonically m -convexity and for $m = 1$, harmonically (s, m) -convexity reduces to harmonically s -convexity in second sense (see [5]) and for $s, m = 1$, harmonically (s, m) -convexity reduces to ordinary harmonically convexity (see [4]).

Proposition 1.15. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a function

- a) if f is (s, m) -convex function in second sense and non-decreasing, then f is harmonically (s, m) -convex function in second sense.
- b) if f is harmonically (s, m) -convex function in second sense and non-increasing, then f is (s, m) -convex function in second sense.

Proof. For all $t \in [0, 1]$, $m \in (0, 1)$ and $x, y \in I$, we have

$$t(1-t)(x - my)^2 \geq 0$$

Hence, the following inequality holds

$$(1.8) \quad \frac{mxy}{mty + (1-t)x} \leq tx + m(1-t)y$$

By the inequality (1.8), the proof is completed. \square

Remark 1.16. According to proposition 1.15, every non-decreasing (s, m) -convex function in second sense is also harmonically (s, m) -convex function in second sense.

Example 1.17. (see[2]) Let $0 < s < 1$ and $a, b, c \in \mathbb{R}$, then function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} a, & x = 0 \\ bx^s + c, & x > 0 \end{cases}$$

is non-decreasing s -convex function in second sense for $b \geq 0$ and $0 \leq c \leq a$. Hence, by proposition 1.15, f is harmonically $(s, 1)$ -convex function.

Proposition 1.18. Let $s \in [0, 1]$, $m \in (0, 1]$, $f : [a, mb] \subset (0, \infty) \rightarrow \mathbb{R}$, be an increasing function and $g : [a, mb] \rightarrow [a, mb]$, $g(x) = \frac{mab}{a+mb-x}$, $a < mb$. Then f is harmonically (s, m) -convex in second sense on $[a, mb]$ if and only if $f \circ g$ is (s, m) -convex in second sense on $[a, mb]$.

Proof. Since

$$(1.9) \quad (f \circ g)(ta + m(1-t)b) = f\left(\frac{mab}{mbt + (1-t)a}\right)$$

for all $t \in [0, 1]$, $m \in (0, 1]$. The proof is obvious from equality (1.9). \square

The following result of the Hermite-Hadamard type holds.

Theorem 1.19. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a harmonically (s, m) -convex function in second sense with $s \in [0, 1]$ and $m \in (0, 1]$. If $0 < a < b < \infty$ and $f \in L[a, b]$, then one has following inequality

$$\frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \min \left[\frac{f(a) + mf(\frac{b}{m})}{s+1}, \frac{f(b) + mf(\frac{a}{m})}{s+1} \right]$$

Proof. Since, $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ is a harmonically (s, m) -convex function in second sense. We have, for $x, y \in I \subset (0, \infty)$

$$f\left(\frac{xy}{ty + (1-t)x}\right) = f\left(\frac{mx \frac{y}{m}}{mt \frac{y}{m} + (1-t)x}\right) \leq t^s f(x) + m(1-t)^s f\left(\frac{y}{m}\right)$$

which gives

$$f\left(\frac{ab}{tb + (1-t)a}\right) \leq t^s f(a) + m(1-t)^s f\left(\frac{b}{m}\right)$$

and

$$f\left(\frac{ab}{ta + (1-t)b}\right) \leq t^s f(b) + m(1-t)^s f\left(\frac{a}{m}\right)$$

for all $t \in [0, 1]$. Integrating on $[0, 1]$ w.r.t 't', we obtain

$$\int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) dt \leq \frac{f(a) + mf(\frac{b}{m})}{s+1}$$

and

$$\int_0^1 f\left(\frac{ab}{ta + (1-t)b}\right) dt \leq \frac{f(b) + mf(\frac{a}{m})}{s+1}$$

However,

$$\int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) dt = \int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) dt = \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx$$

Hence, required inequality is established. \square

Corollary 1.20. *If we take $m = 1$ in theorem 1.19, then we get*

$$\frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{s+1}$$

Corollary 1.21. *If we take $s = 1$ in theorem 1.19, then we get*

$$\frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \min \left[\frac{f(a) + mf(\frac{b}{m})}{2}, \frac{f(b) + mf(\frac{a}{m})}{2} \right]$$

2. MAIN RESULTS

For finding some new inequalities of Hermite-Hadamard type for the functions whose derivatives are harmonically (s, m) -convex in second sense, we need the following lemma

Lemma 2.1. *Let $f : I \subset \mathbb{R}/\{0\} \rightarrow \mathbb{R}$ be a differentiable function on I° with $a < b$. If $f \in L[a, b]$, then*

$$\frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx = \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{(tb + (1-t)a)} f' \left(\frac{ab}{(tb + (1-t)a)} \right) dt$$

Theorem 2.2. *Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, \frac{b}{m} \in I^\circ$ with $a < b$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (s, m) -convex in second sense on $[a, \frac{b}{m}]$ for $q \geq 1$ with $s \in [0, 1]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2^{2-\frac{1}{q}}} [\rho_1(s, q; a, b) |f'(a)|^q + m \rho_2(s, q; a, b) |f'(\frac{b}{m})|^q]^{\frac{1}{q}}$$

where

$$\begin{aligned} \rho_1(s, q; a, b) &= \frac{\beta(1, s+2)}{b^{2q}} {}_2F_1(2q, 1; s+3; 1 - \frac{a}{b}) - \frac{\beta(2, s+1)}{b^{2q}} {}_2F_1(2q, 2; s+3; 1 - \frac{a}{b}) \\ &\quad + \frac{2^{2q-s} \beta(2, s+1)}{(a+b)^{2q}} {}_2F_1(2q, 2; s+3; 1 - \frac{2a}{a+b}) \\ \rho_2(s, q; a, b) &= \frac{\beta(s+1, 2)}{2^s b^{2q}} {}_2F_1(2q, s+1; s+3, \frac{1}{2}(1 - \frac{a}{b})) - \frac{\beta(s+1, 2)}{b^{2q}} {}_2F_1(2q, s+1; s+3, 1 - \frac{a}{b}) \\ &\quad + \frac{\beta(s+2, 1)}{b^{2q}} {}_2F_1(2q, s+2; s+3, 1 - \frac{a}{b}) \end{aligned}$$

β is Euler Beta function defined by

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0$$

and ${}_2F_1$ is hypergeometric function defined by

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad c > b > 0, |z| < 1$$

Proof. From above Lemma and using power mean inequality, we have

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| &\leq \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{(tb + (1-t)a)^2} |f'(\frac{ab}{(tb + (1-t)a)})| dt \\ &\leq \frac{ab(b-a)}{2} \left(\int_0^1 |1-2t| dt \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_0^1 \frac{|1-2t|}{(tb + (1-t)a)^{2q}} |f'(\frac{ab}{(tb + (1-t)a)})|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

Since, $|f'|^q$ is harmonically (s, m) -convex function in second sense, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\int_0^1 \frac{|1-2t| [t^s |f'(a)|^q + m(1-t)^s |f'(\frac{b}{m})|^q]}{(tb + (1-t)a)^{2q}} dt \right)^{\frac{1}{q}} \\ & = \frac{ab(b-a)}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left[|f'(a)|^q \int_0^1 \frac{|1-2t| t^s}{(tb + (1-t)a)^{2q}} dt + m |f'(\frac{b}{m})|^q \int_0^1 \frac{|1-2t| (1-t)^s}{(tb + (1-t)a)^{2q}} dt \right]^{\frac{1}{q}} \\ & = \frac{ab(b-a)}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left[\rho_1(s, q; a, b) |f'(a)|^q + m \rho_2(s, q; a, b) |f'(\frac{b}{m})|^q \right]^{\frac{1}{q}} \end{aligned}$$

It is easy to check that

$$\begin{aligned} \int_0^1 \frac{|1-2t| t^s}{(tb + (1-t)a)^{2q}} dt &= \int_0^{\frac{1}{2}} \frac{|1-2t| t^s}{(tb + (1-t)a)^{2q}} dt + \int_{\frac{1}{2}}^1 \frac{|1-2t| t^s}{(tb + (1-t)a)^{2q}} dt \\ &= \frac{2^{2q-s} \beta(2, s+1)}{(a+b)^{2q}} {}_2F_1(2q, 2; s+3, 1 - \frac{2a}{a+b}) - \frac{\beta(2, s+1)}{b^{2q}} {}_2F_1(2q, 2; s+3, 1 - \frac{a}{b}) \\ &+ \frac{\beta(1, s+2)}{b^{2q}} {}_2F_1(2q, 1; s+3, 1 - \frac{a}{b}) \\ &:= \rho_1(s, q; a, b) \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \frac{|1-2t| (1-t)^s}{(tb + (1-t)a)^{2q}} dt &= \frac{\beta(s+1, 2)}{2^s b^{2q}} {}_2F_1(2q, s+1; s+3, \frac{1}{2}(1 - \frac{a}{b})) - \frac{\beta(s+2, 1)}{b^{2q}} {}_2F_1(2q, s+2; s+3, (1 - \frac{a}{b})) \\ &+ \frac{\beta(s+2, 1)}{b^{2q}} {}_2F_1(2q, s+2; s+3, (1 - \frac{a}{b})) := \rho_2(s, q; a, b) \end{aligned}$$

This completes the proof. \square

If we take $s = m = 1$ in Theorem 2.2, we get the following

Corollary 2.3. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be differentiable function on I° , $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is $(1, 1)$ -harmonically convex in second sense or harmonically convex function on $[a, b]$ for $q \geq 1$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2^{2-\frac{1}{q}}} [\rho_1(1, q; a, b) |f'(a)|^q + \rho_2(1, q; a, b) |f'(b)|^q]^{\frac{1}{q}}$$

Corollary 2.4. If we take $m = 1$ in Theorem 2.2, then we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2^{2-\frac{1}{q}}} [\rho_1(s, q; a, b) |f'(a)|^q + \rho_2(s, q; a, b) |f'(b)|^q]^{\frac{1}{q}}$$

Corollary 2.5. If we take $s = 1$ in Theorem 2.2, we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2^{2-\frac{1}{q}}} [\rho_1(1, q; a, b) |f'(a)|^q + m \rho_2(1, q; a, b) |f'(\frac{b}{m})|^q]^{\frac{1}{q}}$$

Theorem 2.6. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I , $ma, b \in I^\circ$ with $a < b$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (s, m) -convex in second sense on $[a, \frac{b}{m}]$ for $q \geq 1$ with $s \in [0, 1]$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \rho_1^{1-\frac{1}{q}}(0, q; a, b) [\rho_1(s, q; a, b) |f'(a)|^q + m \rho_2(s, q; a, b) |f'(\frac{b}{m})|^q]^{\frac{1}{q}} \end{aligned}$$

Proof. From Lemma, Power mean inequality and harmonically (s, m) -convexity in second sense of $|f'|^q$ on $[a, \frac{b}{m}]$, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \int_0^1 \left| \frac{1-2t}{(tb + (1-t)a)^2} \right| \left| f' \left(\frac{ab}{tb + (1-t)a} \right) \right| dt \\ & \leq \frac{ab(b-a)}{2} \left(\int_0^1 \left| \frac{1-2t}{(tb + (1-t)a)^2} \right| dt \right)^{1-\frac{1}{q}} \left(\int_0^1 \frac{|1-2t|}{(tb + (1-t)a)^2} \left| f' \left(\frac{ab}{tb + (1-t)a} \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{ab(b-a)}{2} \left(\int_0^1 \left| \frac{1-2t}{(tb + (1-t)a)^2} \right| dt \right)^{1-\frac{1}{q}} \left(\int_0^1 \frac{|1-2t| [t^s |f'(a)|^q + m(1-t)^s |f'(\frac{b}{m})|^q]}{(tb + (1-t)a)^2} dt \right)^{\frac{1}{q}} \\ & \leq \frac{ab(b-a)}{2} \rho_1^{1-\frac{1}{q}}(0, q; a, b) [\rho_1(s, q; a, b) |f'(a)|^q + m \rho_2(s, q; a, b) |f'(\frac{b}{m})|^q]^{\frac{1}{q}} \end{aligned}$$

□

Corollary 2.7. *If we take $m = 1$ in Theorem 2.6, then we get*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \rho_1^{1-\frac{1}{q}}(0, q; a, b) [\rho_1(s, q; a, b) |f'(a)|^q + \rho_2(s, q; a, b) |f'(b)|^q]^{\frac{1}{q}} \end{aligned}$$

This is Theorem 1.7 proved by Feixiang Chen and Shanhe Wu in [3].

Corollary 2.8. *If we take $s = 1$ in Theorem 2.6, we get*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \rho_1^{1-\frac{1}{q}}(0, q; a, b) [\rho_1(1, q; a, b) |f'(a)|^q + m \rho_2(1, q; a, b) |f'(\frac{b}{m})|^q]^{\frac{1}{q}} \end{aligned}$$

Corollary 2.9. *If we take $s = m = 1$ in Theorem 2.6, we get*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \rho_1^{1-\frac{1}{q}}(0, q; a, b) [\rho_1(1, q; a, b) |f'(a)|^q + \rho_2(1, q; a, b) |f'(b)|^q]^{\frac{1}{q}} \end{aligned}$$

which is Theorem 1.3 proved by Imdat Iscan in [4].

Theorem 2.10. *Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $ma, b \in I^\circ$ with $a < b$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (s, m) -convex in second sense on $[a, \frac{b}{m}]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ with $s \in [0, 1]$, then*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} [\nu_1(s, q; a, b) |f'(a)|^q + m \nu_2(s, q; a, b) |f'(\frac{b}{m})|^q]^{\frac{1}{q}} \end{aligned}$$

where

$$\nu_1(s, q; a, b) = \frac{\beta(1, s+1)}{b^{2q}} {}_2F_1(2q, 1; s+2, 1 - \frac{a}{b})$$

and

$$\nu_2(s, q; a, b) = \frac{\beta(s+1, 1)}{b^{2q}} {}_2F_1(2q, s+1; s+2, 1 - \frac{a}{b})$$

Proof. From Lemma, Hölder's inequality and harmonically (s, m) -convexity of $|f'|^q$ on $[a, \frac{b}{m}]$, we have

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
& \leq \frac{ab(b-a)}{2} \left(\int_0^1 |1-2t|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \frac{1}{(tb + (1-t)a)^{2q}} \left| f' \left(\frac{ab}{tb + (1-t)a} \right) \right| dt \right)^{\frac{1}{q}} \\
& \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[|f'(a)|^q \int_0^1 \frac{t^s}{(tb + (1-t)a)^{2q}} dt + m |f'(\frac{b}{m})|^q \int_0^1 \frac{(1-t)^s}{(tb + (1-t)a)^{2q}} dt \right]^{\frac{1}{q}} \\
& = \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[|f'(a)|^q \nu_1(s, q; a, b) + m |f'(\frac{b}{m})|^q \nu_2(s, q; a, b) \right]^{\frac{1}{q}}
\end{aligned}$$

where an easy calculation gives

$$\begin{aligned}
\int_0^1 |1-2t|^p dt &= \frac{1}{p+1} \\
\int_0^1 \frac{t^s}{(tb + (1-t)a)^{2q}} dt &= \frac{\beta(1, s+1)}{b^{2q}} {}_2F_1(2q, 1; s+2, 1 - \frac{a}{b}) := \nu_1(s, q; a, b)
\end{aligned}$$

and

$$\int_0^1 \frac{(1-t)^s}{(tb + (1-t)a)^{2q}} dt = \frac{\beta(s+1, 1)}{b^{2q}} {}_2F_1(2q, s+1; s+2, 1 - \frac{a}{b}) := \nu_2(s, q; a, b)$$

This completes the proof. □

Corollary 2.11. *If we take $m = 1$ in Theorem 2.10, then we get*

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
& \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} [\nu_1(s, q; a, b) |f'(a)|^q + \nu_2(s, q; a, b) |f'(b)|^q]^{\frac{1}{q}}
\end{aligned}$$

this is Theorem 1.9 proved by Feixiang Chen and Shanhe Wu in [3].

Corollary 2.12. *If we take $s = 1$ in above Theorem, then we get*

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
& \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} [\nu_1(1, q; a, b) |f'(a)|^q + m \nu_2(1, q; a, b) |f'(\frac{b}{m})|^q]^{\frac{1}{q}}
\end{aligned}$$

Corollary 2.13. *If we take $s = m = 1$ in above Theorem, then we get*

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
& \leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} [\nu_1(1, q; a, b) |f'(a)|^q + \nu_2(1, q; a, b) |f'(b)|^q]^{\frac{1}{q}}
\end{aligned}$$

This is Theorem 1.4 proved by Imdat Iscan in [4].

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